Table 7.9 Gauss-Legendre Abscissas and Weights

$$\int_{-1}^{1} f(x)\,dx = \sum_{k=1}^{N} w_{N,k} f(x_{N,k}) + E_N(f)$$

N	Abscissas, $x_{N,k}$	Weights, $w_{N,k}$	Truncation error, $E_N(f)$
2	-0.5773502692 0.5773502692	1.0000000000 1.0000000000	$\dfrac{f^{(4)}(c)}{135}$
3	± 0.7745966692 0.0000000000	0.5555555556 0.8888888888	$\dfrac{f^{(6)}(c)}{15{,}750}$
4	± 0.8611363116 ± 0.3399810436	0.3478548451 0.6521451549	$\dfrac{f^{(8)}(c)}{3{,}472{,}875}$
5	± 0.9061798459 ± 0.5384693101 0.0000000000	0.2369268851 0.4786286705 0.5688888888	$\dfrac{f^{(10)}(c)}{1{,}237{,}732{,}650}$
6	± 0.9324695142 ± 0.6612093865 ± 0.2386191861	0.1713244924 0.3607615730 0.4679139346	$\dfrac{f^{(12)}(c)2^{13}(6!)^4}{(12!)^3 13!}$
7	± 0.9491079123 ± 0.7415311856 ± 0.4058451514 0.0000000000	0.1294849662 0.2797053915 0.3818300505 0.4179591837	$\dfrac{f^{(14)}(c)2^{15}(7!)^4}{(14!)^3 15!}$
8	± 0.9602898565 ± 0.7966664774 ± 0.5255324099 ± 0.1834346425	0.1012285363 0.2223810345 0.3137066459 0.3626837834	$\dfrac{f^{(16)}(c)2^{17}(8!)^4}{(16!)^3 17!}$

Theorem 7.9 (Gauss-Legendre Three-Point Rule). If f is continuous on $[-1, 1]$, then

(17) $$\int_{-1}^{1} f(x)\,dx \approx G_3(f) = \frac{5f(-\sqrt{3/5}) + 8f(0) + 5f(\sqrt{3/5})}{9}.$$

The Gauss-Legendre rule $G_3(f)$ has degree of precision $n = 5$. If $f \in C^6[-1, 1]$, then

(18) $$\int_{-1}^{1} f(x)\,dx = \frac{5f(-\sqrt{3/5}) + 8f(0) + 5f(\sqrt{3/5})}{9} + E_3(f),$$

where

(19) $$E_3(f) = \frac{f^{(6)}(c)}{15,750}.$$

Table 3.1.1. First-order Forward and Backward Difference Representations.

	f_{n-4}	f_{n-3}	f_{n-2}	f_{n-1}	f_n	f_{n+1}	f_{n+2}	f_{n+3}	f_{n+4}
<u>Forward</u>									
$h.f'(x_n)$ =					-1	1			
$h^2.f''(x_n)$ =					1	-2	1		
$h^3.f'''(x_n)$ =					-1	3	-3	1	
$h^4.f^{iv}(x_n)$ =					1	-4	6	-4	1
<u>Backward</u>									
$h.f'(x_n)$ =				-1	1				
$h^2.f''(x_n)$ =			1	-2	1				
$h^3.f'''(x_n)$ =		-1	3	-3	1				
$h^4.f^{iv}(x_n)$ =	1	-4	6	-4	1				

Table 3.1.2. Second-order Central Difference Representations.

	f_{n-2}	f_{n-1}	f_n	f_{n+1}	f_{n+2}
$2h.f'(x_n)$ =		-1	0	1	
$h^2.f''(x_n)$ =		1	-2	1	
$2h^3.f'''(x_n)$ =	-1	2	0	-2	1
$h^4f^{iv}(x_n)$ =	1	-4	6	-4	1

Table 3.1.3. Second-order Forward and Backward Difference Representations.

	f_{n-5}	f_{n-4}	f_{n-3}	f_{n-2}	f_{n-1}	f_n	f_{n+1}	f_{n+2}	f_{n+3}	f_{n+4}	f_{n+5}
<u>Forward</u>											
$2h.f'(x_n)$ =						-3	4	-1			
$h^2.f''(x_n)$ =						2	-5	4	-1		
$2h^3.f'''(x_n)$ =						-5	18	-24	14	-3	
$h^4.f^{iv}(x_n)$ =						3	-14	26	-24	11	-2
<u>Backward</u>											
$2h.f'(x_n)$ =				1	-4	3					
$h^2.f''(x_n)$ =			-1	4	-5	2					
$2h^3.f'''(x_n)$ =		3	-14	24	-18	5					
$h^4.f^{iv}(x_n)$ =	-2	11	-24	26	-14	3					

Table 3.1.4. Fourth-order Central Difference Representations.

	f_{n-3}	f_{n-2}	f_{n-1}	f_n	f_{n+1}	f_{n+2}	f_{n+3}
$12h.f'(x_n)$ =		1	-8	0	8	-1	
$12h^2.f''(x_n)$ =		-1	16	-30	16	-1	
$8h^3.f'''(x_n)$ =	1	-8	13	0	-13	8	-1
$6h^4.f^{iv}(x_n)$ =	-1	12	-39	56	-39	12	-1

خلاصه روش رانگ کوتای مرتبه سوم:

$$y'(x_i) = \frac{dy}{dx} = f(x_i\,, y_i)$$

$$k_1 = hf(x_i\,,\ y_i)$$

$$k_2 = hf\left(x_i + \frac{h}{2}\,,\ y_i + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_i + h\,,\ y_i + 2k_2 - k_1)$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

خلاصه روش رانگ کوتای مرتبه چهارم:

$$y'(x_i) = \frac{dy}{dx} = f(x_i\,, y_i)$$

$$k_1 = hf(x_i\,,\ y_i)$$

$$k_2 = hf\left(x_i + \frac{h}{2}\,,\ y_i + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_i + \frac{h}{2}\,,\ y_i + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_i + h\,,\ y_i + k_3)$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

خلاصه روش اویلر:

$$y'(x_i) = \frac{dy}{dx} = f(x_i, y_i)$$

$$y_{i+1} = y_i + h\,f(x_i, y_i)$$

خلاصه روش رانگ کوتای مرتبه دوم:

$$y'(x_i) = \frac{dy}{dx} = f(x_i\,,\ y_i)$$

$$k_1 = hf(x_i\,,\ y_i)$$

$$k_2 = hf(x_i + h\,,\ y_i + k_1)$$

$$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)$$